Measure and Orbit Equivalence of Graph Products

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https://arxiv.org/abs/2401.04635

1 Measure Equivalence

1.1 Motivations: classification criteria

Classification up to isomorphism

➤ G and H are isomorphic iff $\exists$ a countable set $\Omega \neq \emptyset$, st. $G, H \circlearrowleft \Omega$

- freely;
- the 2 actions commute;
- and are both transitive.

Classification up to quasi-isometry [Gromov]

➤ G QI H iff $\exists$ locally compact space Ω st. $G, H \circlearrowleft \Omega$

- properly discontinuously[1]
- the 2 actions commute;
- and both admit a compact fundamental domain[2].

Gromov introduced measure equivalence
as a measured analogue of quasi-isometry.

[1] **properly discontinuous:** for all $K \subseteq X$ compact, $|\{g \in G : gK \cap K \neq \emptyset\}| < +\infty$.
[2] A **fundamental domain** is a subset of Ω that contains exactly one element of each orbit.

1.2 Definitions

Definition 1.1

We say that G and H are measure equivalent if there exists a measure space (Ω, m) st $G, H \circlearrowleft \Omega$

- freely, measure preservingly;
- the 2 actions commute;
- and both admit a fundamental domain of *finite* measure.

Ex. Two lattices[3] in the same locally compact group

Definition 1.2

G and H are orbit equivalent if they are measure equivalent with a common fundamental domain.

Ex. $F_2 \overset{ME}{\sim} F_3$ but not OE [Gaboriau].

Proposition 1.3 (*Furmann-Gaboriau*)

G and H are OE iff there exists a probability space (X, μ) st $G, H \circlearrowleft X$ freely, measure preservingly and the two actions have the same orbits.

1.3 Examples

Flexibility behaviour

Theorem [Ornstein-Weiss] All amenable groups are ME to $\mathbb{Z}$.

[3] **Lattices:** discrete subgroups of finite covolume

Rigidity behaviour

Furman, '99 Let H be a lattice in $PSL_3(\mathbb{R})$.
If $G \overset{ME}{\sim} H$ then G is is commensurable up to finite kernel[4] to another lattice in $PSL_3(\mathbb{R})$.

Kida, '06 If $G \overset{ME}{\sim}$ Mapping Class Group[5]
then G is commensurable (up to finite kernel) to it.

Guirardel-Horbez, '21 If G ME $\mathrm{Out}(F_n)$,
then G is virtually $\mathrm{Out}(F_n)$. ($n \geqslant 3$)

2 Graph products

2.1 Definition

Definition 2.1

Let Γ be a finite graph and let $\{G_v\}_{\in\Gamma}$ be a family of groups. The **graph product** G_Γ is

$$G_\Gamma := *_{v\in V\Gamma} G_v / \langle\langle [g,h] \mid g \in G_v,\ h \in G_w,\ (v,w) \in E\Gamma \rangle\rangle,$$
$$:= \langle G_v\, v \in V\Gamma \mid [G_v, G_w]\, \forall (v,w) \in E\Gamma \rangle.$$

[4] G and H are **commensurable** if they have finite index subgroups $G_1 \leqslant G$ and $H_1 \leqslant H$ that are isomorphic. They are **commensurable up to finite kernel** if they have finite index subgroups $G_1 \leqslant G$ and $H_1 \leqslant H$ such that $G_1/N_G \simeq H_1/N_H$ where $N_G \leqslant G_1$ and $N_H \leqslant H_1$ are finite normal subgroups.

[5] The **Mapping class group** of a connected, closed, orientable surface S is the group $\mathrm{Mod}(S) := \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$, where
$\mathrm{Homeo}^+(S)$ is the group of orientation-preserving homeo of S;
It's endowed w/ the topology induced by $\delta(f,g) := \sup_{x\in S} d(f(x), g(x))$.
$\mathrm{Homeo}_0(S)$ is the connected component of the identity for this topology. It is the set of homeo of S isotopic to the identity.

Example 2.2.

- If Γ has no edges, then $G_\Gamma = *_{v\in V\Gamma} G_v$.
- If Γ complete, then $G_\Gamma = \times_{v\in V\Gamma} G_v$.
- If $G_v = \mathbb{Z}/2\mathbb{Z}$ for all v, G is a RACG.
- If $G_v = \mathbb{Z}$ for all v, G is a RAAG.

2.2 RAAGs and RACGs

Let W_n be the RACG over the n-gon[6].

For all $n, m \geqslant 5 \quad W_n$ and W_m are ME.

Let A_1, A_2 be RAAG with finite $\mathrm{Out}(A_i)$.

Theorem 2.3 (*Horbez-Huang, '21*)

$A_1 \overset{ME}{\sim} A_2$ iff $\Gamma_1 \simeq \Gamma_2$ iff $A_1 \simeq A_2$.

→ *This matches the QI classification*[7]*!*

Theorem 2.4 (*Huang, '17*)

Let A_1, A_2 be RAAG with finite $\mathrm{Out}(A_i)$.

$$A_{\Gamma_1} \text{ QI } A_{\Gamma_2} \quad \text{iff} \quad \Gamma_1 \simeq \Gamma_2 \quad \text{iff} \quad A_1 \simeq A_2.$$

[6] W_n acts properly, cocompactly on $\mathbb{H}^2_{\mathbb{R}}$ as the group generated by the reflections over a right-angled hyperbolic n-gon. Therefore all groups W_n are cocompact discrete subgroup of $PSL_2(\mathbb{R})$ (cocompact Fuchsian), so they are all ME (in fact commensurable).

[7] This is specific to the finite Out case. There are examples of right-angled Artin groups which are quasi-isometric but not measure equivalent. For instance, the groups $G_n = (F_3 \times F_3) * F_n$ are all quasi-isometric but non pariwise ME.

➤ Goal: Extend this classification results to graph products.

Finite Outer automorphism group Let Γ finite simple graph and $v \in V\Gamma$. Denote $\mathrm{lk}(v) := \{\text{neighbours of } v\}$.

We say that

- Γ is *transvection free*, ie. Γ does not contain 2 distinct vertices v and w s.t. $\mathrm{lk}(v) \subseteq B_\Gamma(w,1)$.
- for all $v \in V\Gamma$ the induced subgraph $\Gamma \backslash B_\Gamma(v,1)$ is connected. We will say that Γ does not have any *partial conjugations*.

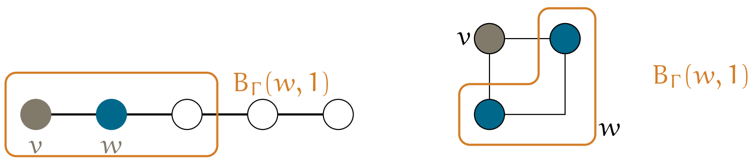

Figure 1: Graphs with transvections

Criterion [Laurence-Servatius] $\mathrm{Out}(A_\Gamma)$ is finite iff Γ has no transvection, no partial conjugation.

Example 2.5. If Γ is an n-gon with $n \geqslant 5$, then A_Γ has finite Out.

Example 2.6 ($\mathrm{Out}(F_n)$)**.** If Γ contains at least 2 vertices and has no edges, then $\mathrm{lk}(v) = \emptyset$ for all $v \in V\Gamma$. Therefore, $\mathrm{lk}(v) \subset B(w,1)$ for all $v \neq w$. Hence Γ has transvections and therefore $\mathrm{Out}(A_\Gamma)$ is infinite. Recall that if Γ has no edges then A_Γ is a free group. We thus just checked that $\mathrm{Out}(F_n)$ is infinte ($n \geqslant 2$).

Remark 2.7 (Why this terminology?). The names (having transvections or partial conjugations) comes from the corresponding automorphisms one can define in this setting. More precisely: Since we are considering RAAGs, $G_v = \mathbb{Z}$ for all $v \in V\Gamma$. Denote by $s_v = 1_{G_v}$ the standard generator of G_v.

- If there exists $v \neq w \in V\Gamma$ st. $\mathrm{lk}(v) \subseteq B_\Gamma(w,1)$, then one can define $\varphi \in \mathrm{Aut}(A_\Gamma)$ by letting $\varphi(s_v) := s_v s_w$ and $\varphi(s_u) = s_u$ for all $u \neq v$. Such an automorphism is called a transvection.
- If there exists $v \in V\Gamma$ st. the induced subgraph $\Gamma \backslash B_\Gamma(v,1)$ is disconnected. Let C be one connected component of $\Gamma \backslash B_\Gamma(v,1)$. Then one can define $\varphi \in \mathrm{Aut}(A_\Gamma)$ by letting $\varphi(s_w) = s_v s_w s_v^{-1}$ for all $w \in C$ and $\varphi(s_u) = s_u$ for all $u \notin C$. Such an automorphism is called a partial conjugation.

2.3 ME classification of graph products

Theorem 2.8 (*E.-Horbez '24*)

If $|V\Gamma|, |V\Lambda| \geqslant 2$ and

➤ Γ, Λ transvection free, have no partial conjugat°;

➤ G_v, H_w are countably infinite $\forall v \in \Gamma$, $w \in \Lambda$;

then,

$G_\Gamma \overset{ME}{\sim} H_\Lambda$;

$\Leftrightarrow$ $G_\Gamma \overset{OE}{\sim} H_\Lambda$;

$\Leftrightarrow$ There exists a graph isomorphism $\theta : \Gamma \to \Lambda$ st. $G_v \overset{OE}{\sim} H_{\theta(v)}$ for all $v \in V\Gamma$.

Example 2.9. The two graph products from figure 2 are not measure equivalent.

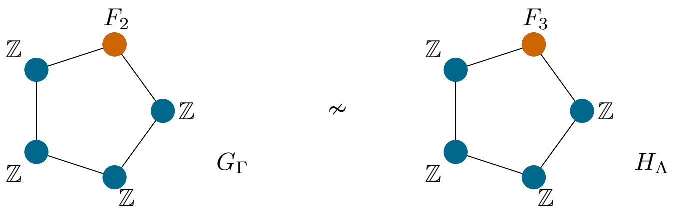

Figure 2: Two graph products that are not measure equivalent

Remark 2.10. We actually show that every measure equivalence coupling is an orbit equivalence coupling.

Corollary 2.11 (*EH, '24*)

Under the same hypotheses

G_Γ and H_Λ are commensurable;
$\Leftrightarrow$ G_Γ and H_Λ are strongly commensurable;
$\Leftrightarrow$ There exists a graph isoM $\theta : \Gamma \to \Lambda$ st.
G_v and $H_{\theta(v)}$ strongly commensurable, for all $v \in V\Gamma$.

G and H are **strongly commensurable** if the isomorphic finite index subgroups G_1 and H_1 have the same finite index.

Remark 2.12. The proof of this corollary relies on the following criterion: Two countable groups G and H are commensurable if and only if there exists a non-empty countable set Ω equipped with commuting free actions of G and H such that each actions admits a finite fundamental domain.

In particular commensurable implies QI and commensurable up to finite kernel implies ME.

3 Some tools

3.1 Extension graphs

- **Extension graph** of G_Γ: the graph Γ^e_G such that

$$V\Gamma^e_G := \{gG_vg^{-1} \mid g \in G,\ v \in V\Gamma_G\},$$
$E\Gamma^e_G$ vertices linked by an edge iff they commute

- G acts on Γ^e_G by conjugation.
  A fundamental domain is the subgraph spanned by $\{G_v : v \in V\Gamma\}$.

Theorem 3.1 (*E.-Horbez*)

If $|V\Gamma|, |V\Lambda| \geqslant 2$ and

- Γ, Λ transvection free, have no partial conjugat°;
- G_v, H_w are countably infinite $\forall v \in \Gamma$, $w \in \Lambda$, then

$$G_\Gamma \overset{ME}{\sim} H_\Gamma \quad \Rightarrow \quad \Gamma^e_G \simeq \Gamma^e_H .$$

3.2 Strategy

Show that $G \overset{ME}{\sim} H \Rightarrow G \overset{OE}{\sim} H$
Goal: Find a common fundamental domain

1. Work w/ actions we understand $\to$ **Extension graph** Γ^e_G, Γ^e_H
2. Show: $G \overset{ME}{\sim} H \Rightarrow \Gamma^e_G \simeq \Gamma^e_G$
3. Find an equivariant map $\theta : \Omega \to \mathrm{Isom}(\Gamma^e_G, \Gamma^e_H)$.
4. Find a common fundamental domain $Y \subset \mathrm{Isom}(\Gamma^e_G, \Gamma^e_H)$.
5. $X := \theta^{-1}(Y)$ is a common fundamental domain in Ω.