Quantitative Measure Equivalence and Graph Products

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(Joint work with Camille Horbez)

Slides and written notes

Link to the article

https://s.42l.fr/AEConf

Main result

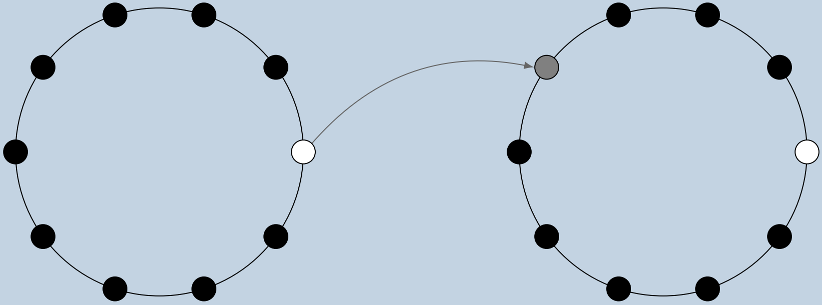
Theorem. [E.–Horbez, '24] Assume that

- Γ, Λ have no transvection, no partial conjugation;
- G_v, H_w are infinite f.g. groups $\forall v \in \Gamma,\ w \in \Lambda$.

If $G_\Gamma \overset{(\varphi,\psi)}{\sim} H_\Lambda$,

Then there exists a graph isomorphism $\theta : \Gamma \to \Lambda$ st. $G_v \overset{(\varphi,\psi)}{\sim} H_{\theta(v)}$ for all $v \in V\Gamma$.

I — Measure Equivalence



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  - and both admit a *compact* fundamental domain.

A **fundamental domain** is a subset of Ω that contains exactly one element of each orbit.

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Ex. If $\mathsf{H} \leqslant \mathsf{G}$ is of finite index, then $\mathsf{G} \overset{\mathrm{ME}}{\sim} \mathsf{H}$.

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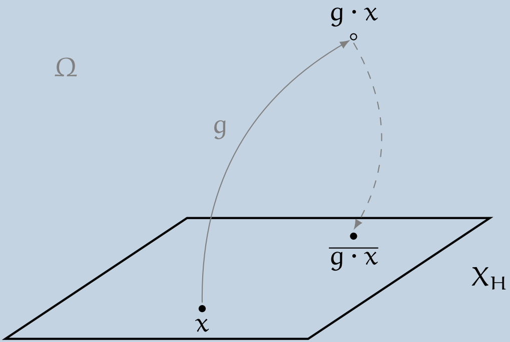
→ **Refine** this relation to **distinguish** groups with different geometries.

II — Quantitative Measure Equivalence



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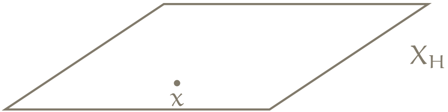

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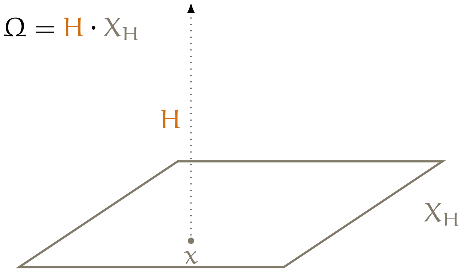

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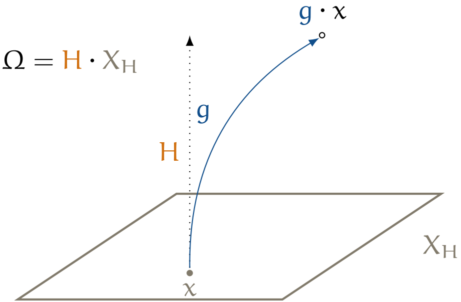

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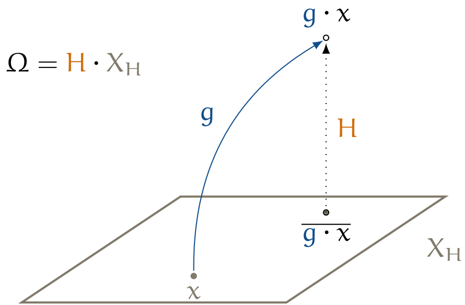

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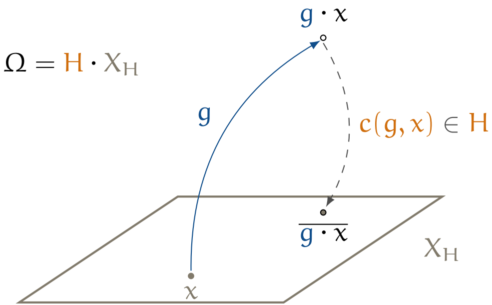

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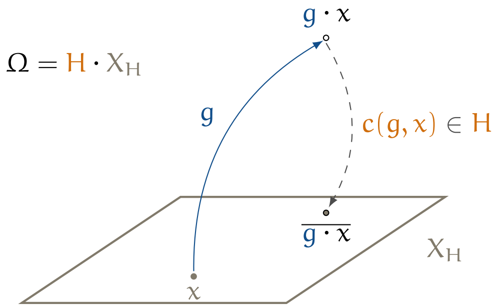

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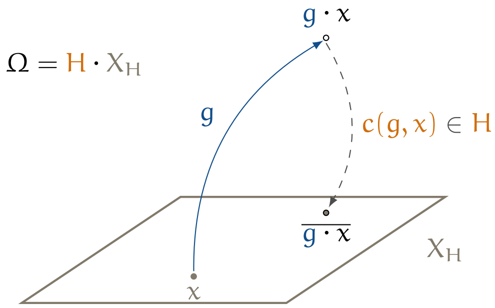

When is $x \mapsto |c(g, x)|_{S_H}$ bounded? In L^p?

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Definition. We have an (L^p, L^q)**-integrable** ME coupling from G to H if $\forall g \in G$, $\forall h \in H$

$$\int_{X_H} \left(|c(g,x)|_{S_H} \right)^p d\mu < \infty \quad \int_{X_G} \left(|c'(h,x)|_{S_G} \right)^q d\mu < \infty.$$

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Let $\varphi,\psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be two unbounded increasing functions.

[Delabie, Koivisto, Le Maître, Tessera, '20]

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$$\int_{X_H} \varphi\left(\delta_g |c(g,x)|_{S_H}\right) d\mu < \infty \quad \int_{X_G} \psi\left(\delta_h |c'(h,x)|_{S_G}\right) d\mu < \infty.$$

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Abbreviation. $G \overset{(\varphi,\psi)}{\sim} H$.

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Rk. Extended to (φ, L^0)-ME, w/ φ subadditive by [DKLMT].

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Rk: Extended to locally compact groups by **Paucar**.

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Rk: **no obstruction for non-amenable groups** (yet)...

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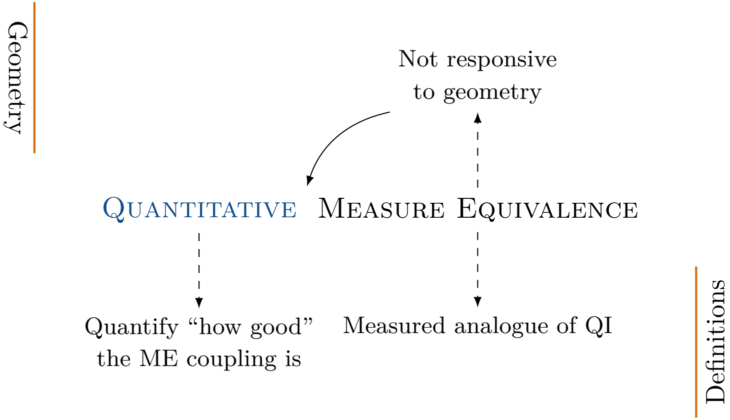
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Quantify "how good" the ME coupling is
Measured analogue of QI
Definitions

Recap

Geometry

Behaves well with respect to geometry

Not responsive to geometry

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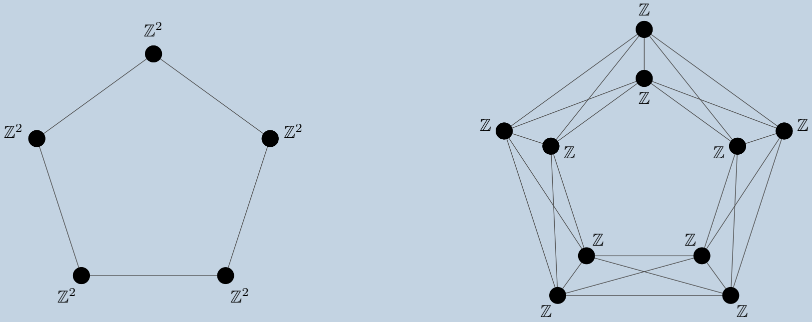
Quantify "how good" the ME coupling is

Measured analogue of QI

Definitions

III — Graph Products



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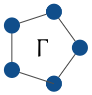

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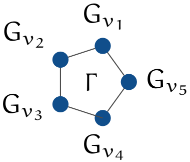

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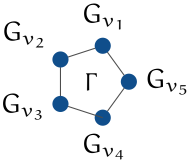

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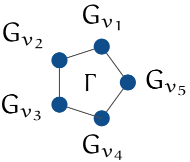

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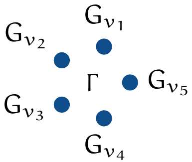

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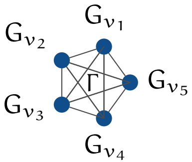

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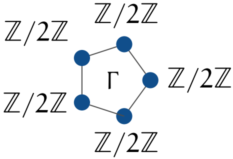

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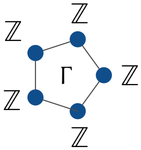

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What happens to graph products? What about the quantification?

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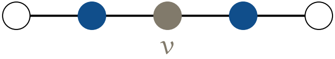

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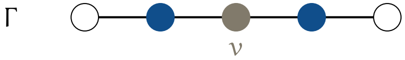

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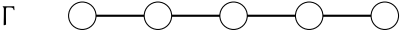

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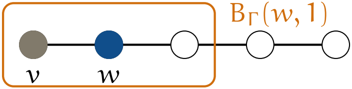
Γ 

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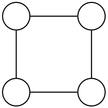
Γ 

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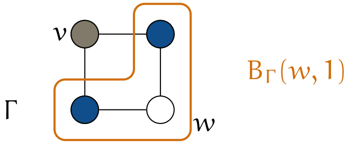

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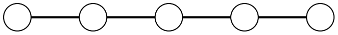

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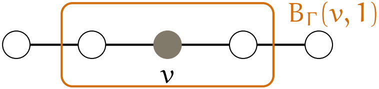
Γ 

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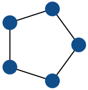

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Rk. This gives the 1st obstruction in the non-amenable world.

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IV — A word about the proofs

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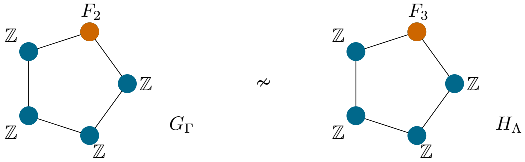

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Show that $G_v \overset{(\varphi,\psi)}{\sim} H_v$ for all $v \in V\Gamma \Rightarrow G_\Gamma \overset{(\varphi,\psi)}{\sim} H_\Gamma$.

Let

$V\Gamma = \{v_1, \dots, v_n\}$	$X := Z \times Y_1 \times \cdots \times Y_n$
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Action of G_Γ : $\forall g \in G_\Gamma$ and a.e. $(z, y_1, \dots, y_n) \in X$

$$g \cdot (z, y_1, \dots, y_n) = (\qquad\qquad).$$

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$$h_i \cdot (z, y_1, \dots, y_n) := c'_i(h_i, y_i) \cdot (z, y_1, \dots, y_n).$$

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$$h_i \cdot (z, y_1, \dots, y_n) := c_i'(h, y_i) \cdot (z, y_1, \dots, y_n).$$

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Action of $\mathbf{H}_{\mathbf{v_i}}$: Let $c_i' : H_{\nu_i} \times Y_i \to G_{\nu_i}$ and $h_1 \in H_{\nu_1}$

$$h_1 \cdot (z, y_1, \dots, y_n) := \left(c_1'(h, y_1) \cdot z, c_1'(h, y_1) \cdot y_1, y_2, \dots, y_n\right).$$

Extends to an action of H Let $(\nu_1, \nu_2) \in E\Gamma$ and $h_i \in H_{\nu_i}$,

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$=$

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$$\begin{aligned} & h_2 \cdot (h_1 \cdot (z, y_1, \dots, y_n)) \\ = & h_2 \cdot \left(c'_1(h, y_1) \cdot z, c'_1(h, y_1) \cdot y_1, y_2, \dots, y_n\right), \\ = & \end{aligned}$$

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$$\begin{aligned} & h_2 \cdot (h_1 \cdot (z, y_1, \dots, y_n)) \\ =& h_2 \cdot \left(c_1'(h, y_1) \cdot z, c_1'(h, y_1) \cdot y_1, y_2, \dots, y_n\right), \\ =& \left(c_2'(h_2, y_2) c_1'(h_1, y_1) \cdot z, c_1'(h, y_1) \cdot y_1, c_2'(h_2, y_2) \cdot y_2, y_3, \dots, y_n\right), \end{aligned}$$

Show that $G_v \overset{(\varphi,\psi)}{\sim} H_v$ for all $v \in V\Gamma \Rightarrow G_\Gamma \overset{(\varphi,\psi)}{\sim} H_\Gamma$.

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$$h_2 \cdot (h_1 \cdot (z, y_1, \ldots, y_n))$$
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$$= c_1'(h_1, y_1) \cdot \left(c_2'(h_2, y_2) \cdot z, y_1, c_2'(h_2, y_2) \cdot y_2, \dots, y_n\right),$$

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=&c_1'(h_1, y_1) \cdot \left(c_2'(h_2, y_2) \cdot z, y_1, c_2'(h_2, y_2) \cdot y_2, \dots, y_n\right), \\
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& h_2 \cdot (h_1 \cdot (z, y_1, \dots, y_n)) \\
=& \left(c_2'(h_2, y_2) c_1'(h_1, y_1) \cdot z, c_1'(h_1, y_1) \cdot y_1, c_2'(h_2, y_2) \cdot y_2, \dots, y_n\right), \\
=& c_1'(h_1, y_1) \cdot \left(c_2'(h_2, y_2) \cdot z, y_1, c_2'(h_2, y_2) \cdot y_2, \dots, y_n\right), \\
=& h_1 \cdot \left(c_2'(h_2, y_2) \cdot z, y_1, c_2'(h_2, y_2) \cdot y_2, \dots, y_n\right), \\
=& h_1 \cdot h_2 \cdot (z, y_1, y_2, \dots, y_n),
\end{aligned}$$

$$g \cdot (z, y_1, \dots, y_n) = (g \cdot z, r_1(g) \cdot y_1, \dots, r_n(g) \cdot y_n)$$
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Let $S_G := \sqcup_{v \in V\Gamma} S_{G_v}$

$$g \cdot (z, y_1, \dots, y_n) = (g \cdot z, r_1(g) \cdot y_1, \dots, r_n(g) \cdot y_n)$$
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Let $S_G := \sqcup_{v \in V\Gamma} S_{G_v}$ $\qquad S_H := \sqcup_{v \in V\Gamma} S_{H_v}$.

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$\Rightarrow$ **φ-integrable.**

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By definition of the action of H_{v_i}

$$s_i' \cdot x = \underbrace{c_i'(s_i', y_i)}_{\in G_{v_i}} \cdot x$$
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$$\Rightarrow |c'(s_i', x)|_G = |c_1'(h_1, y_1)|_G = |c_1'(h_1, y_1)|_{G_{v_i}}.$$

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Appendix

Non-quantitative statement

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$$G_\Gamma \overset{\text{ME}}{\sim} H_\Lambda;$$

$$\Leftrightarrow G_\Gamma \overset{\text{OE}}{\sim} H_\Lambda;$$

$\Leftrightarrow$ There exists a graph isoM $\theta : \Gamma \to \Lambda$ st. $G_v \overset{\text{OE}}{\sim} H_{\theta(v)}$ for all $v \in V\Gamma$.