Measure and orbit equivalence of graph products

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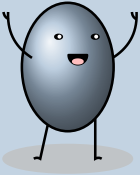
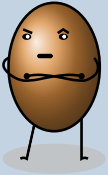
Université Lyon 1
Joint work with Camille Horbez

https ://s.42l.fr/AEConf

I — Measure Equivalence





Measured Dynamics

Geometric group theory

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  - properly discontinuously ;
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  - and both admit a *compact* fundamental domain.

A **fundamental domain** is a subset of Ω that contains exactly one element of each orbit.

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Theorem. [Ornstein-Weiss, '80]
All infinite countable amenable groups are OE to $\mathbb{Z}$.

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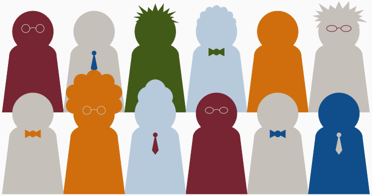

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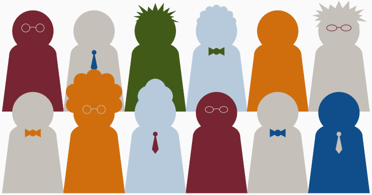

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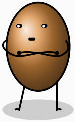

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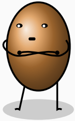

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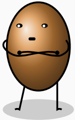

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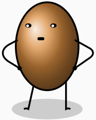
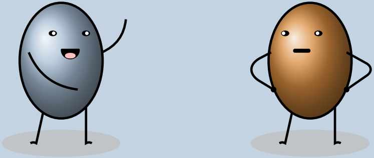


- **[Guirardel-Horbez, '21]**
  If $G \overset{ME}{\sim} \mathrm{Out}(F_n)$ $(n \geqslant 3)$
  Then G is virtually isomorphic to $\mathrm{Out}(F_n)$.

II — Graph Products

and RAAGs, and RACGs



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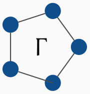

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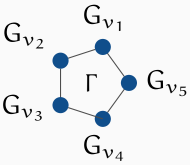

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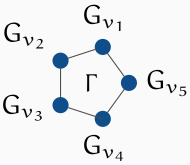

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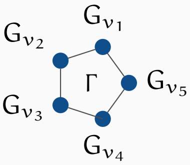

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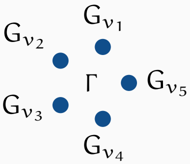

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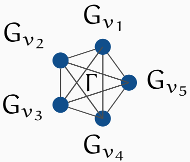

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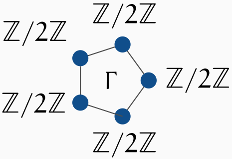

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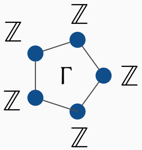

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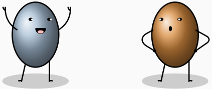

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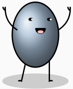

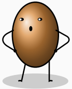

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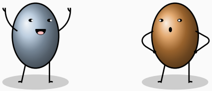

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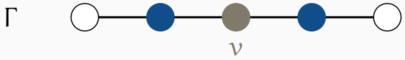

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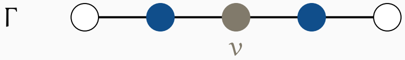

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$\mathrm{Out}(A_\Gamma)$ is finite
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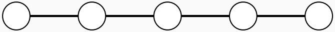

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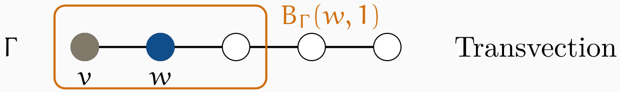

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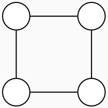

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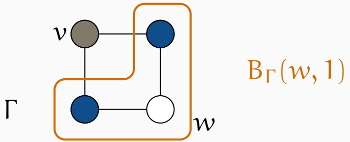

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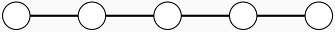

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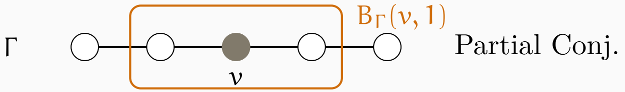

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iff Γ has no transvection, no partial conjugation.

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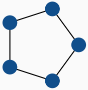

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II.3 — ME and graph products

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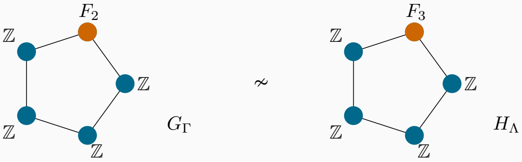

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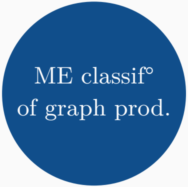

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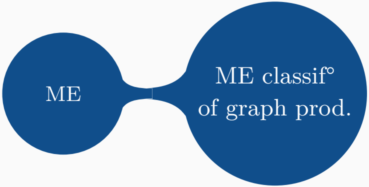

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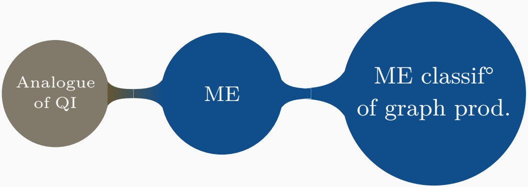

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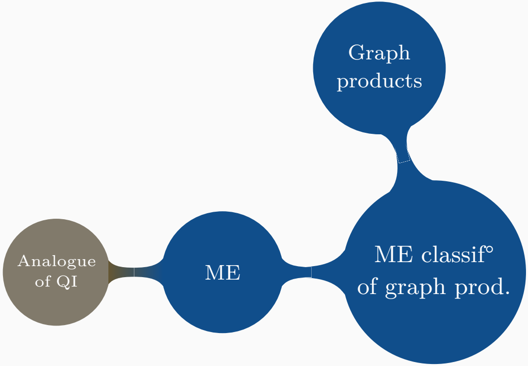

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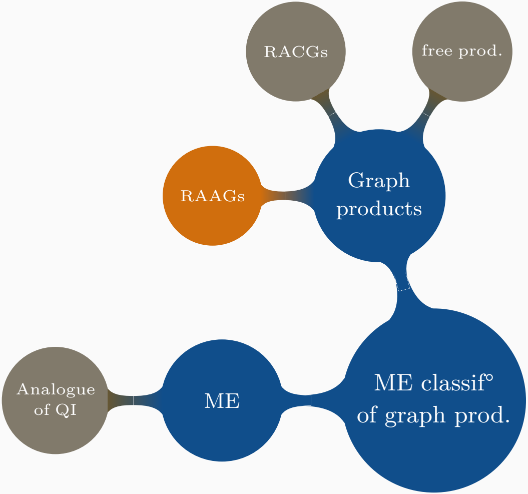

RACGs
free prod.
RAAGs
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ME classif°
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Analogue
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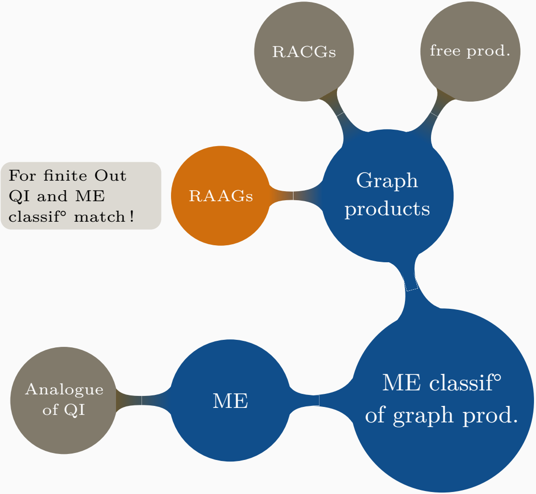
Summary



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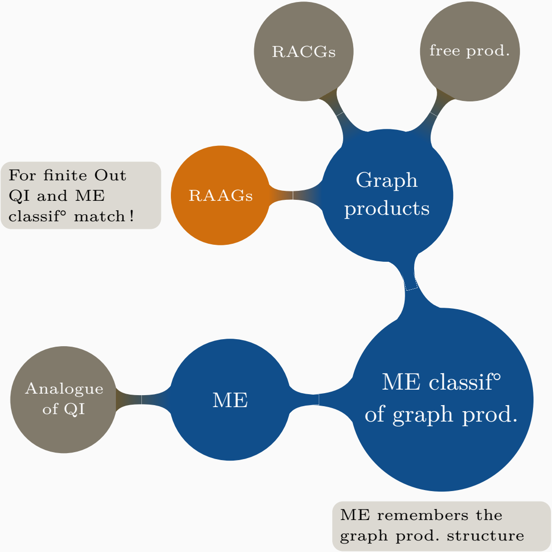
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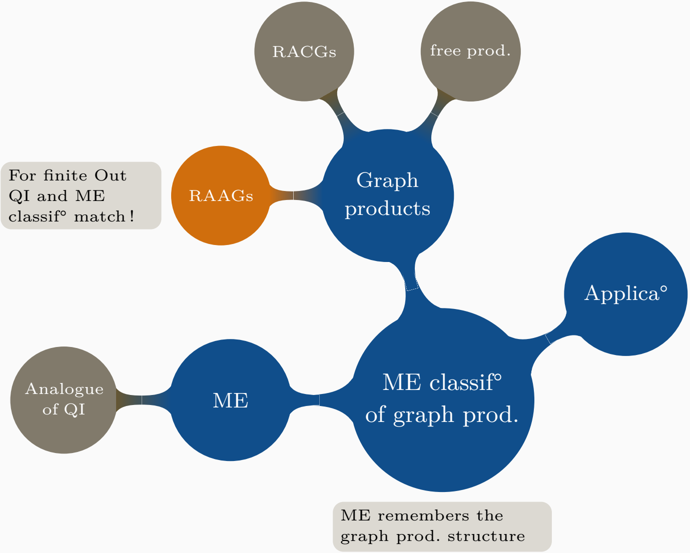
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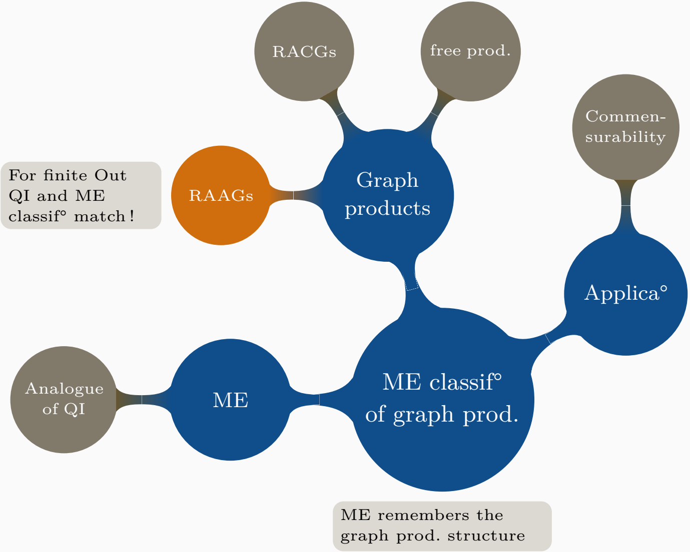
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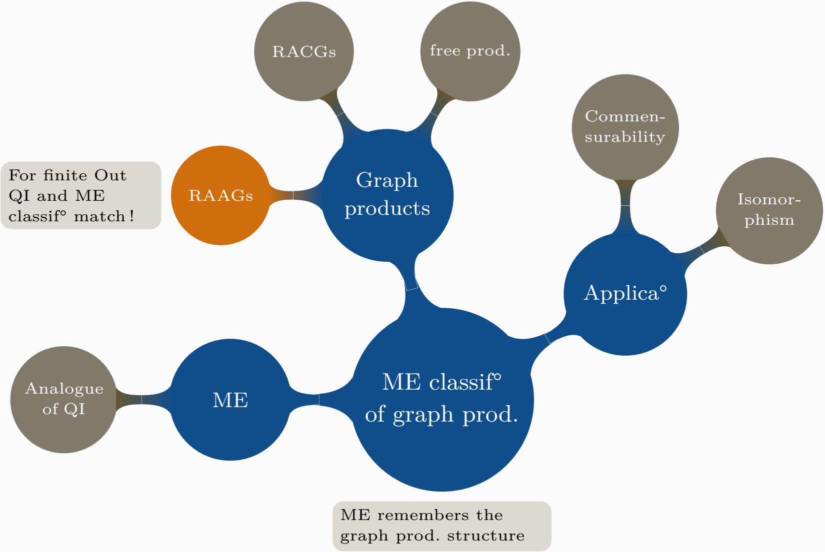
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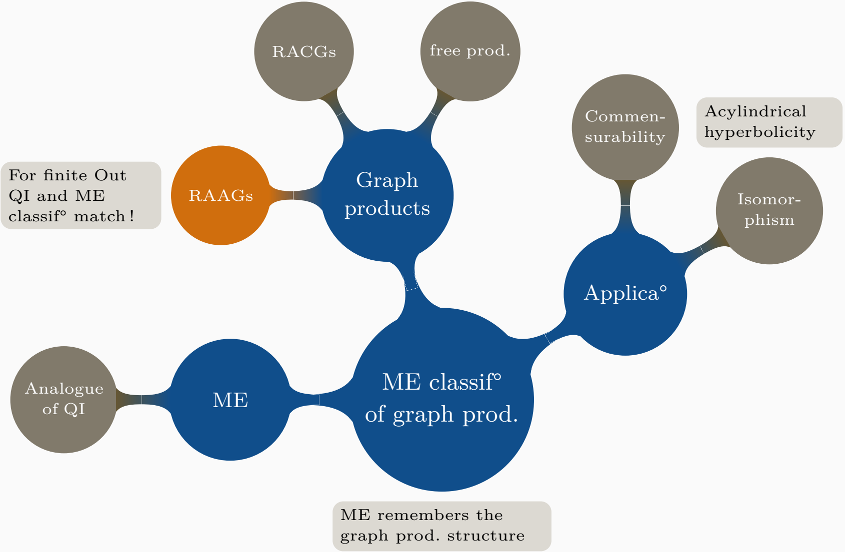

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III — Tools

When geometry meets measured groupoids

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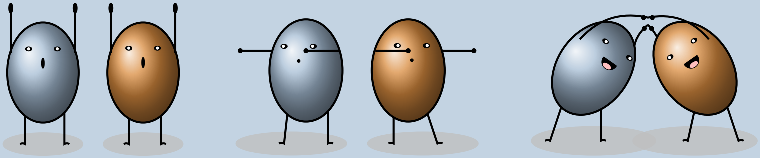

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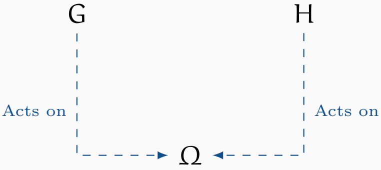

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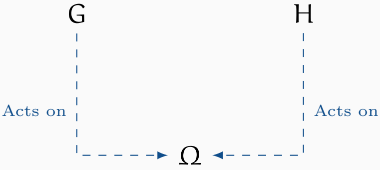

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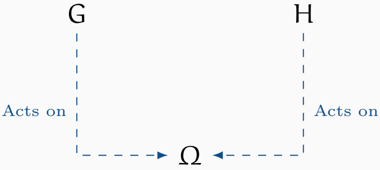

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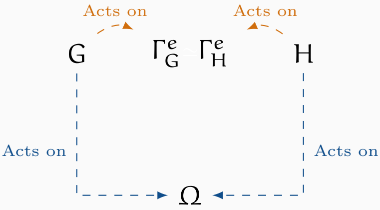

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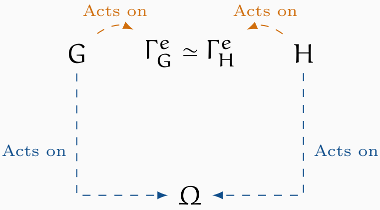

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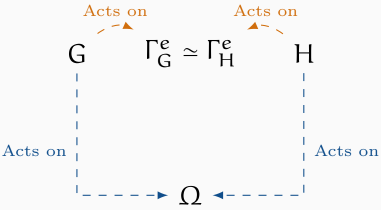
$\mathrm{Isom}(\Gamma^e_{\mathsf{G}}, \Gamma^e_{\mathsf{H}})$



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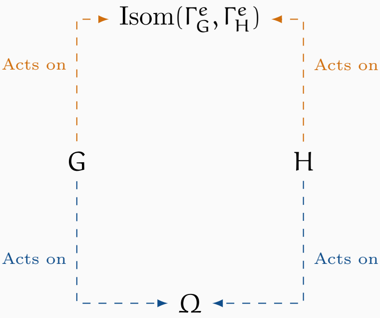

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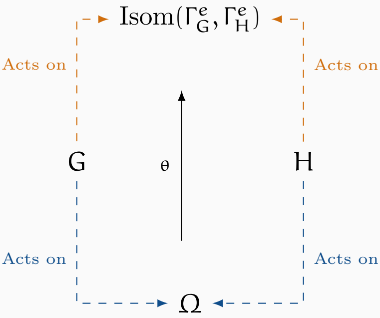

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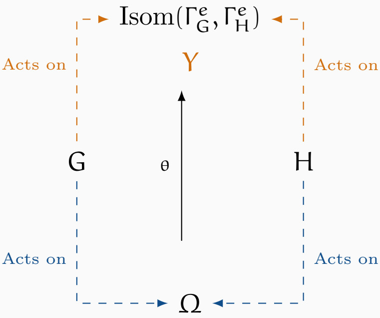

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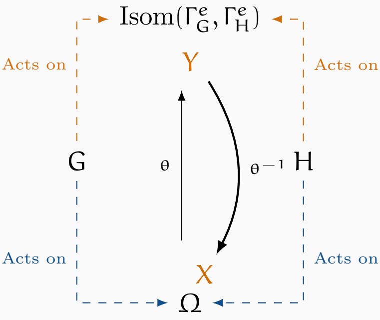
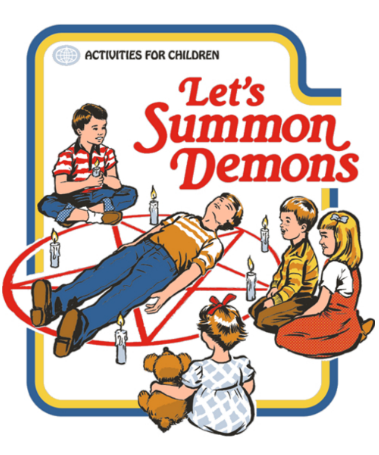
5. $X := \theta^{-1}(Y)$ is a common fundamental domain in Ω.



Appendix

Vertex type groupoids



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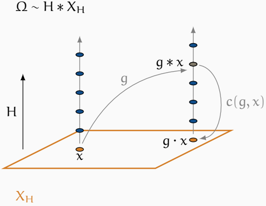
  $G \curvearrowright X_H$ *via* $\{g \cdot x\} = X_H \cap (H * g * x)$.



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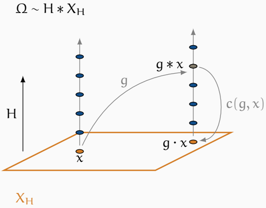

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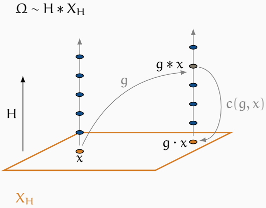

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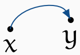

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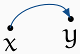

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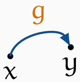

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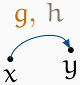

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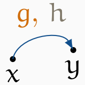

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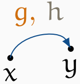

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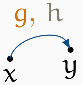

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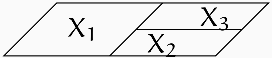

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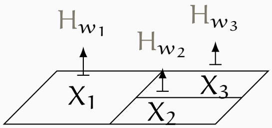

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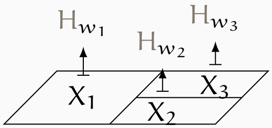

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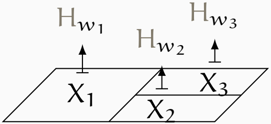

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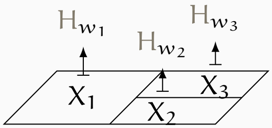

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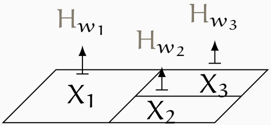

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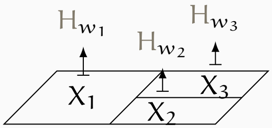

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- Define a map $\theta : X \to \mathrm{Isom}(\Gamma_G, \Gamma_H)$ st

$$\theta(x) :\ v \in V\Gamma_G \to w_i \quad \text{if } x \in X_i.$$